Expresar $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ en coordenadas polares.

Solución:

La relación entre las coordenadas cartesianas y las polares es:

$$\begin{cases} x = r\cos\phi \\ y = r\phi \end{cases} \rightarrow \begin{cases} x^2+y^2 = r^2 \\ \frac{y}{x} = \tan\phi \end{cases}$$

por lo que las derivadas se pueden calcular aplicando la regla de la cadena

$$\frac{\partial \Phi}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial \Phi}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial \Phi}{\partial \phi}$$

donde

$$\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2}}{\partial x} = \frac{1}{2}\left(x^2+y^2\right)^{-1/2}\, 2x = \frac{x}{r} = \cos\phi$$

análogamente $\frac{\partial\phi}{\partial x}$ se puede obtener derivando en la segunda ecuación

$$\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{\partial}{\partial x}\tan\phi \rightarrow -\frac{y}{x^2} = \frac{1}{\cos^2\phi}\frac{\partial\phi}{\partial x}$$

de donde se obtiene

$$\frac{\partial\phi}{\partial x} = -\frac{y}{x^2}\cos^2\phi = -\frac{\phi}{r}$$

que permite calcular

$$\frac{\partial\Phi}{\partial x} = \cos\phi\frac{\partial\Phi}{\partial r} - \frac{\phi}{r}\frac{\partial\Phi}{\partial\phi}$$

y

$$\frac{\partial^2\Phi}{\partial x^2} = \left(\cos\phi\frac{\partial}{\partial r} + -\frac{\phi}{r}\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial\Phi}{\partial r} + -\frac{\phi}{r}\frac{\partial\Phi}{\partial\phi}\right)$$

$$= \cos^2\phi\frac{\partial^2\Phi}{\partial r^2} - \phi\cos\phi\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}\left(\frac{\Phi}{r}\right) - \frac{\phi}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\phi}\left(\cos\phi\,\Phi\right) + \frac{\phi}{r^2}\frac{\partial}{\partial\phi}\left(\phi\frac{\partial\Phi}{\partial\phi}\right)$$

Análogamente

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \phi$$

y

$$\frac{1}{x} = \frac{1}{\cos^2\phi}\frac{\partial\phi}{\partial y} \rightarrow \frac{\partial\phi}{\partial y} = \frac{1}{x}\cos^2\phi = \frac{\cos\phi}{r}$$

de donde se obtiene

$$\frac{\partial\Phi}{\partial y} = \phi\frac{\partial\Phi}{\partial r} + \frac{\cos\phi}{r}\frac{\partial\Phi}{\partial\phi}$$

y

$$\frac{\partial^2\Phi}{\partial y^2} = \left(\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}\right)\left(\phi\frac{\partial\Phi}{\partial r} + \frac{\cos\phi}{r}\frac{\partial\Phi}{\partial\phi}\right)$$

$$=^2\phi\frac{\partial^2\Phi}{\partial r^2} + \phi\cos\phi\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}\left(\frac{\Phi}{r}\right) + \frac{\cos\phi}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\phi}\left(\phi\,\Phi\right) + \frac{\cos\phi}{r^2}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial\Phi}{\partial\phi}\right)$$

de tal forma que

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi = \frac{\partial^2\Phi}{\partial r^2} - \frac{\phi}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\phi}\left(\cos\phi\,\Phi\right) + \frac{\cos\phi}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\phi}\left(\phi\,\Phi\right)$$

$$+\frac{\phi}{r^2}\frac{\partial}{\partial\phi}\left(\phi\,\frac{\partial\Phi}{\partial\phi}\right) + \frac{\cos\phi}{r^2}\frac{\partial}{\partial\phi}\left(\cos\phi\,\frac{\partial\Phi}{\partial\phi}\right)$$

$$= \frac{\partial^2\Phi}{\partial r^2} - \frac{\phi}{r}\frac{\partial}{\partial r}\left(-\phi\Phi + \cos\phi\,\frac{\partial\Phi}{\partial\phi}\right) + \frac{\cos\phi}{r}\frac{\partial}{\partial r}\left(\cos\phi\Phi + \phi\,\frac{\partial\Phi}{\partial\phi}\right)$$

$$+\frac{\phi}{r^2}\left(\cos\phi\,\frac{\partial\Phi}{\partial\phi} + \phi\,\frac{\partial^2\Phi}{\partial\phi^2}\right) + \frac{\cos\phi}{r^2}\left(-\phi\,\frac{\partial\Phi}{\partial\phi} + \cos\phi\,\frac{\partial^2\Phi}{\partial\phi^2}\right)$$

$$= \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}$$